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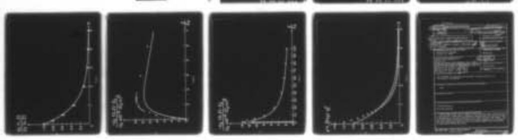

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PROJECT SQUID

TECHNICAL REPORT UC-2-PU

AXIAL FLOW IN TRAILING LINE VORTICES

BY

MAHINDER S. UBEROI
BHIMSEN K. SHIVAMOGGI and SIN-SUNG CHEN
UNIVERSITY OF COLORADO
BOULDER, COLORADO 80309

PROJECT SQUID HEADQUARTERS
CHAFFEE HALL
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907

JUNE 1978

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Technical Report UC-2-PU

PROJECT SQUID

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Mahinder S. Uberoi
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University of Colorado
Boulder, Colorado 80309

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ABSTRACT

Axial flow in the core of laminar steady trailing vortex from the tip of a semi-infinite lifting wing is analyzed, assuming that the pressure gradient is determined by the swirl velocities of an ideal infinite line vortex. The axial and lateral variations of the axial velocity depend on the strength of the vortex and initial axial velocity distribution which must be specified at some station behind the wing.

I. INTRODUCTION

Vortices occur naturally and are also of technical interest. As an example, consider the trailing vortex which is formed near the tip of a semi-infinite lifting wing in a uniform steady flow u_o. The vorticity shed near the tip of the wing rapidly rolls up into a concentrated core along u_o and the flow becomes axisymmetric, such that at lateral distance $r \to \infty$ the velocity $u_\theta\ (r,z) \to \Gamma_o/2\pi r$ where θ refers to the angular coordinate and z is the axial distance along the vortex axis, and Γ_o is the circulation around the vortex. In the core the swirl velocity decreases with downstream distance z. The pressure is constant at large r and hence the pressure increases along z and in the core there must be an axial velocity difference $u_z(r,z)$ from the prevailing velocity u_o. These simple considerations show the existence of an axial velocity difference between the core and the surrounding fluid. The theory is needed to find its detailed distributions and dependence on various parameters. This paper is concerned with flows which are laminar everywhere.

Newman[1] has calculated the axial velocity variations in the vortex core by assuming that both u_θ and u_z are small compared with u_o and the equations governing u_θ and u_z become uncoupled and may be solved separately. Dosanjh[2] et al have measured the velocities in the turbulent core of a vortex and compared the results with Newman's calculation for the laminar case assuming that the laminar kinematic voscosity may be replaced by an equivalent turbulent exchange coefficient.

Batchelor[3] has analyzed the axial velocity in the vortex core which is caused by the axial pressure gradient associated with the decaying vortex. In effect, he assumes that u_θ is independent of u_z but the later is influenced by the former. We will examine at the end of this paper the validity of this approximation. Betchelor's analysis has been criticized by Tam[4] which in turn has been criticized by Herron[5]. We show that all three authors have missed some essential points. We give here the correct analysis.

II. GOVERNING EQUATIONS

The equation governing u_θ in trailing vortex under boundary layer (slender vortex) approximation is[6]

$$\frac{\partial}{\partial z}(u_o + u_z)u_\theta r^2 + \frac{\partial}{\partial r}(u_r u_\theta r^2) = \nu \frac{\partial}{\partial r} \; r^3 \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \tag{1}$$

where ν is the kinematic viscosity.

All other investigators have invariably assumed that the axial velocity difference $u_z\,(r,z)$ between the core and the surrounding, and the radial velocity $u_r\,(r,z)$ may be neglected, which we may call LINE VORTEX approximation. Presently we use these approximations and subsequently we examine the limits of their validity and estimate the effect of the neglected terms on the dynamics of the vortex flow. For the line vortex, Eq. (1) becomes

$$u_o \frac{\partial}{\partial z} \; u_\theta r^2 = \nu \frac{\partial}{\partial r} \; r^3 \frac{\partial}{\partial r} \; \frac{u_\theta}{r} \tag{2}$$

which has a self-similar solution

$$u_\theta(r,z) 2\pi r = \Gamma_o(1-e^{-\eta}) \tag{3}$$

where

$$\eta = u_o r^2/4\nu z \tag{4}$$

Assuming that u_0 is given by Eq. (3) we calculate u_z in the following approximate manner. We assume that the radial pressure gradient is given approximately by the equation.

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} \tag{5}$$

Using Eqs. (3) - (5) we can calculate axial pressure gradient[3], thus

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = (\frac{\Gamma_o}{2\pi})^2 \frac{u_o}{8\nu z^2} \quad (P\eta)' \tag{6}$$

where

$$p(\eta) = \int_\eta^\infty \frac{(1-e^{-t})^2}{t^2} \, dt \tag{7}$$

The linearized equation of motion for u_z is

$$u_o \frac{\partial}{\partial z} u_z = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) u_z \tag{8}$$

III. AXIAL VELOCITY

The axial velocity difference u_z is determined by the initial distribution of u_z and by the pressure gradient along the vortex axis which is due to decaying u_θ in the vortex core. The diffusion Eq. (8) has a unique solution for the sources given by Eq. (6) and an initial condition $u_z^*(r,z_o)$ prescribed at a station z_o. Using standard Green Function technique, see for example Luikov[7], the solution due to the sources is given by

$$\frac{u_z(r,z)}{u_o} = \frac{-1}{64\pi^2}\left(\frac{\Gamma_o}{\nu}\right)^2 \int_{t_o}^{t} \int_0^\infty \frac{\{P(\eta)\eta\}'}{t'^2} \frac{e^{-\left[\frac{s^2+s'^2}{4(t-t')}\right]}}{(t-t')} I_o\left(\frac{ss'}{2(t-t')}\right) s'ds'dt' \tag{9}$$

where $s = ru_o/\nu$, $t = u_o z/\nu$, $\eta = s^2/4t$ and $t_o = u_o z_o/\nu$ and I_o is the modified Bessel function.

The contribution $u_z^*(r,z)$ due to the initial $u_z^*(r,z_o)$ is again obtained using standard methods by the relation

$$\frac{u_z^*(r,z)}{u_o} = \tfrac{1}{2}\int_0^\infty u_z^*(s,t_o) \frac{e^{-\left[\frac{s^2+s'^2}{4(t-t_o')}\right]}}{t-t_o} I_o\left(\frac{ss'}{2(t-t_o)}\right) s'ds' \tag{10}$$

$$\simeq \int_0^\infty u_z^*(r,z_o)2\pi r dr \cdot \frac{1}{4\pi\nu z} e^{\frac{-r^2u_o}{4\nu z}} \quad \text{for } z \gg z_o \tag{11}$$

In the present notation total axial velocity is the sum of $u_z(r,z)$ and $u_z^*(r,z)$. The sources given by (6) have a non-integrable singularity at $z = 0$, so that z_o cannot be zero. The initial velocity cannot be prescribed at $z = 0$, and therefore the solution is not independent of z_o .

Unfortunately, Eq. (9) cannot be integrated in terms of known functions. We have numerically integrated it for $z_o u_o/\nu = 1$ and $z/z_o = 10$, 100, and 1,000. We have also integrated it for another initial station $z_o u_o/\nu = 10$ and for $z/z_o = 10$, and 100. The velocity profile $u_z(r,z)/u_z(o,z)$ are shown in Fig. 1. The maximum non-demensional velocity defect $u_z(o,z)/u_o$, as a function of z, is shown in Fig. 2 and 3.

IV. AXIAL VELOCITY USING EQUIVALENT SOURCES

The function $\left[P(\eta)\eta\right]$ of the source distribution is shown in Fig. 4 together with $e^{-\eta}$. The area under the two curves (i.e., the integral of sources in the radial plane) is the same and the curves are also similar. Therefore, we may approximate Eq. (8) by

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) = -\left(\frac{\Gamma_o}{2\pi}\right)^2 \frac{u_o}{8\nu z^2} e^{\frac{-r^2 u_o}{4\nu z}} \tag{12}$$

Using (12) as the source Eq. (8) has the solution

$$\frac{u_z(r,z)}{u_o} = \frac{-1}{u_o}\left(\frac{\Gamma_o}{2\pi}\right)^2 \frac{1}{8\nu z} \ln z/z_o e^{\frac{-r^2 u_o}{4\nu z}} \tag{13}$$

We now find that

$$u_z(r,z)/u_z(o,z) = e^{\frac{-r^2 u_o}{4\nu z}} \tag{14}$$

and

$$u_z(o,z)/u_o = \left(\frac{\Gamma_o}{2\pi}\right)^2 \frac{1}{8\nu z} \ln z/z_o \tag{15}$$

The results given by Eqs. (14) and (15) compare favorably with the numerical computation as shown in Fig. 1, 2 and 3, respectively. We note that u_z is the axial velocity defect due to the pressure gradient in the vortex core starting with uniform axial velocity at z_o. We may add to it the contribution of any initial axial velocity excess or defect using Eq. (10) or Eq. (11) as appropriate.

A brief version of this analysis was given earlier.[8]

V. DISCUSSION

A solution of Eq. (8) for the sources given by Eq. (6) was given by Batchelor [3] which was criticized for non-uniqueness by Tam[4] which in turn was criticized by Herron[5]. The difficulties are due to the singularity of the source distribution at $z = 0$, see Eq. (6), which no one has noticed. If the source distribution were integrable at $z = 0$ then it might have been possible to get a solution which does not depend on the detailed distribution of sources and which would be valid for large z.

We can estimate the neglect of radial velocity and the variations in axial velocity in Eq. (2) using the procedure given in reference 6. Integrating Eq. (1) we have

$$u_o\int_o^\infty \frac{\partial}{\partial z} u_\theta r^2 dr + \int_o^\infty \frac{\partial}{\partial z} u_\theta u_z r2dr + \left[ru_r\right]^\infty \cdot \frac{\Gamma_o}{2\pi} = -\nu\Gamma_o/\pi \qquad (16)$$

or using the continuity equation and $\Gamma = 2\pi r u_\theta$, we have

$$\underset{\text{convection}}{u_o\int_o^\infty \frac{\partial}{\partial z} \Gamma\, rdr = \int_o^\infty \frac{\partial}{\partial z} (\Gamma_o-\Gamma)u_z rdr} \quad \underset{\text{diffusion}}{- 2\nu\Gamma_o} \qquad (17)$$

The ratio of the neglected terms which represent the radial and the associated axial convection of angular momentum to the diffusion term, which is retained, can be calculated from the analytical solution for u_z based on equivalent sources.

Thus we find

$$\int_o^\infty \frac{\partial}{\partial z} (\Gamma_o-\Gamma)u_z rdr/-2\nu\Gamma_o = \left(\frac{\Gamma_o}{\nu}\right) (\Gamma_o/64\pi z_o u_o) \qquad (18)$$

We note that for large z the axial velocity due to any initial distribution does not contribute to the above ratio. The approximations used to replace the trailing vortex by a line vortex are valid if the above ratio is much less than unity.

The comparison of this theory with available experiment can only be made when Eq. (18) is satisfied and the flow is laminar. We do not know of any such experiments. In general, the results of analysis based on laminar flow cannot be applied to the flow in the turbulent state. In the case at hand, it is not possible to replace the kinematic viscosity by a turbulent exchange coefficient and use the theoretical results presented here for comparison with experimental results for turbulent trailing vortices. The reasons are not directly relevant here, they are quite extensive and are presented elsewhere.[9]

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Fig. 1. Axial velocity defect distribution as a function η,
- $e^{-\eta}$; x $z/z_o = 1{,}000$, $z_o = 1$; Δ $z/z_o = 100$, $z_o = 1$;
0 $z/z_1 = 10$, $z_0 = 1$; $\square$ $z/z_o = 100$, $z_o = 10$, + $z/z_o = 10$

Fig. 2. Axial velocity defect distribution on the vortex axis for z/z_o from 1 to 7; - approximate solution using equivalent sources; x numerical solution using exact distribution of sources;

Fig 3. Axial Velocity defect distribution on the vortex axis for z/z_o from 1 to 100; - approximate solution using equivalent sources; x numerical solution using exact distribution of sources;

Fig. 4. Comparison of sources with equivalent sources; x sources distribuiton; - equivalent source distribution.

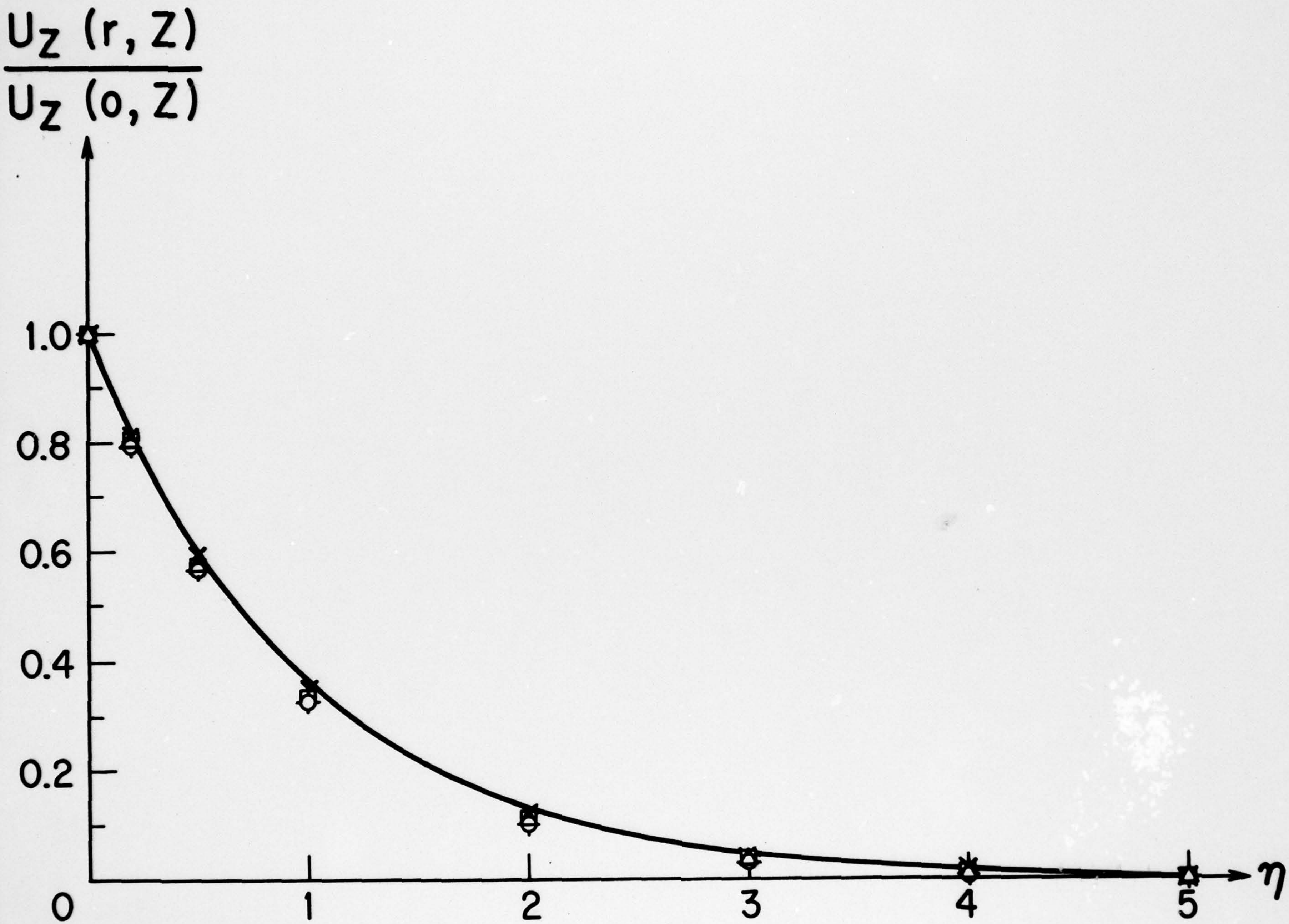
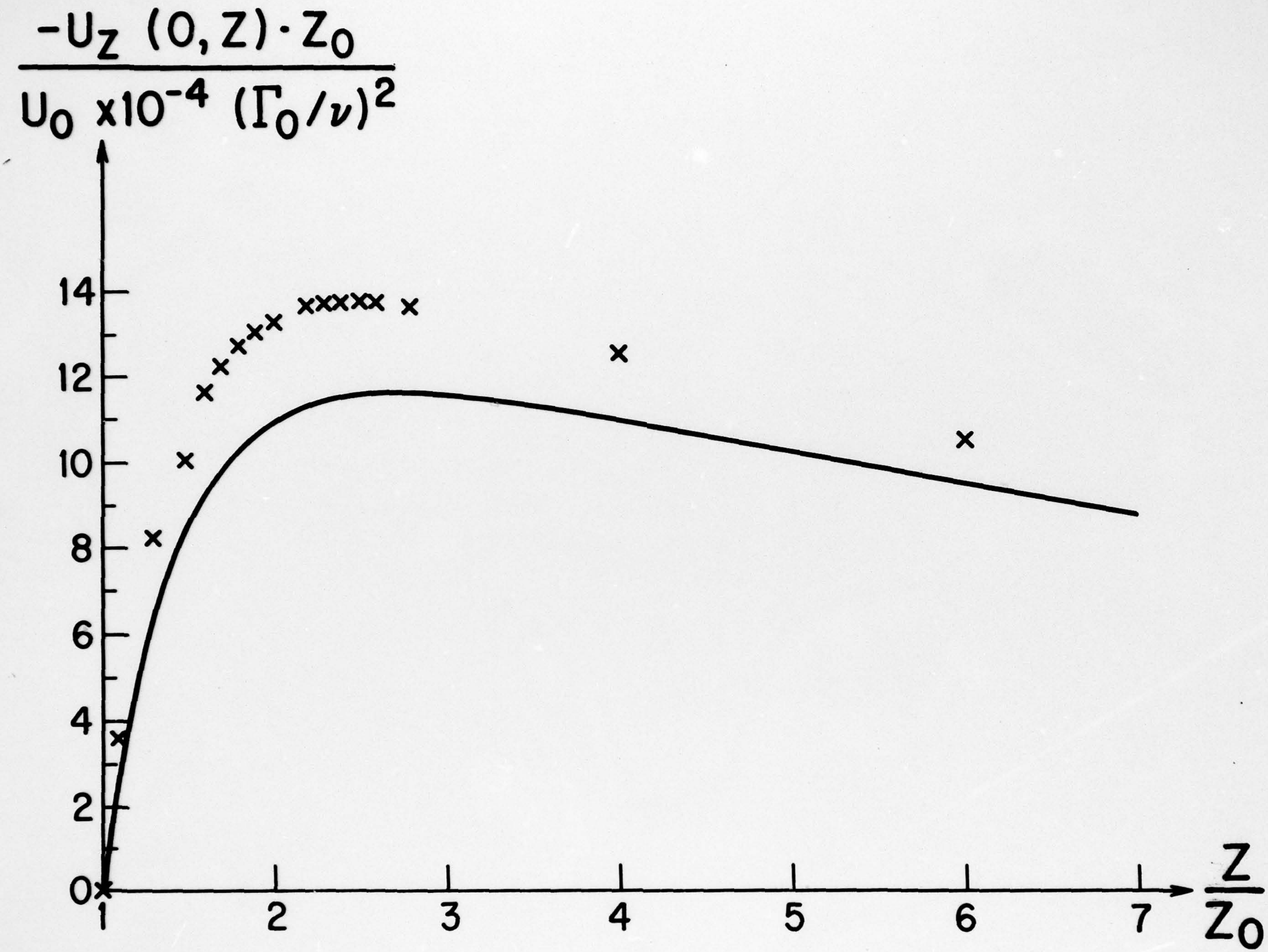


Figure 1


$\frac{-U_Z\,(0,Z)\cdot Z_0}{U_0 \times 10^{-4}\,(\Gamma_0/\nu)^2}$
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$\frac{Z}{Z_0}$

Figure 2

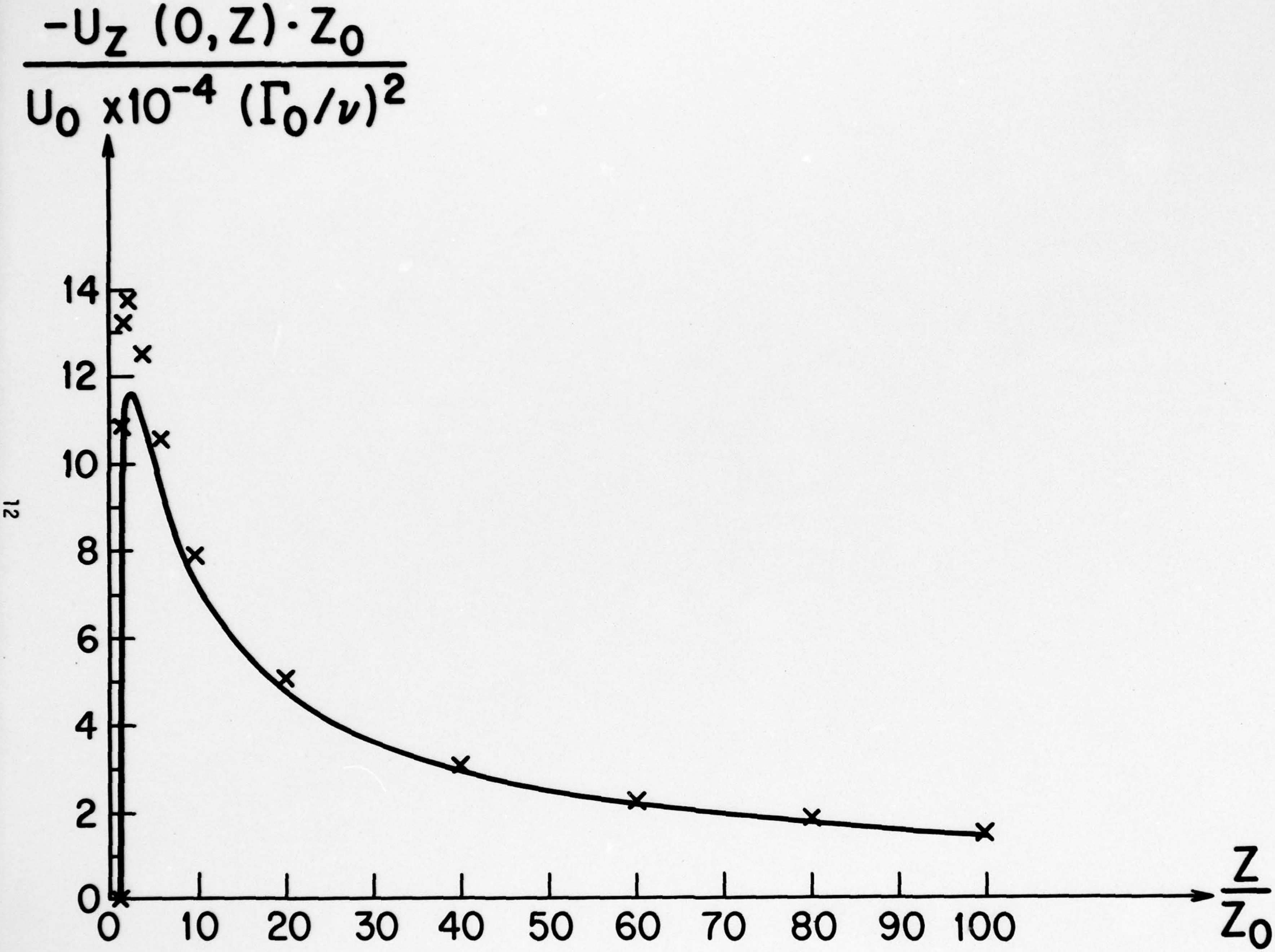
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Figure 3

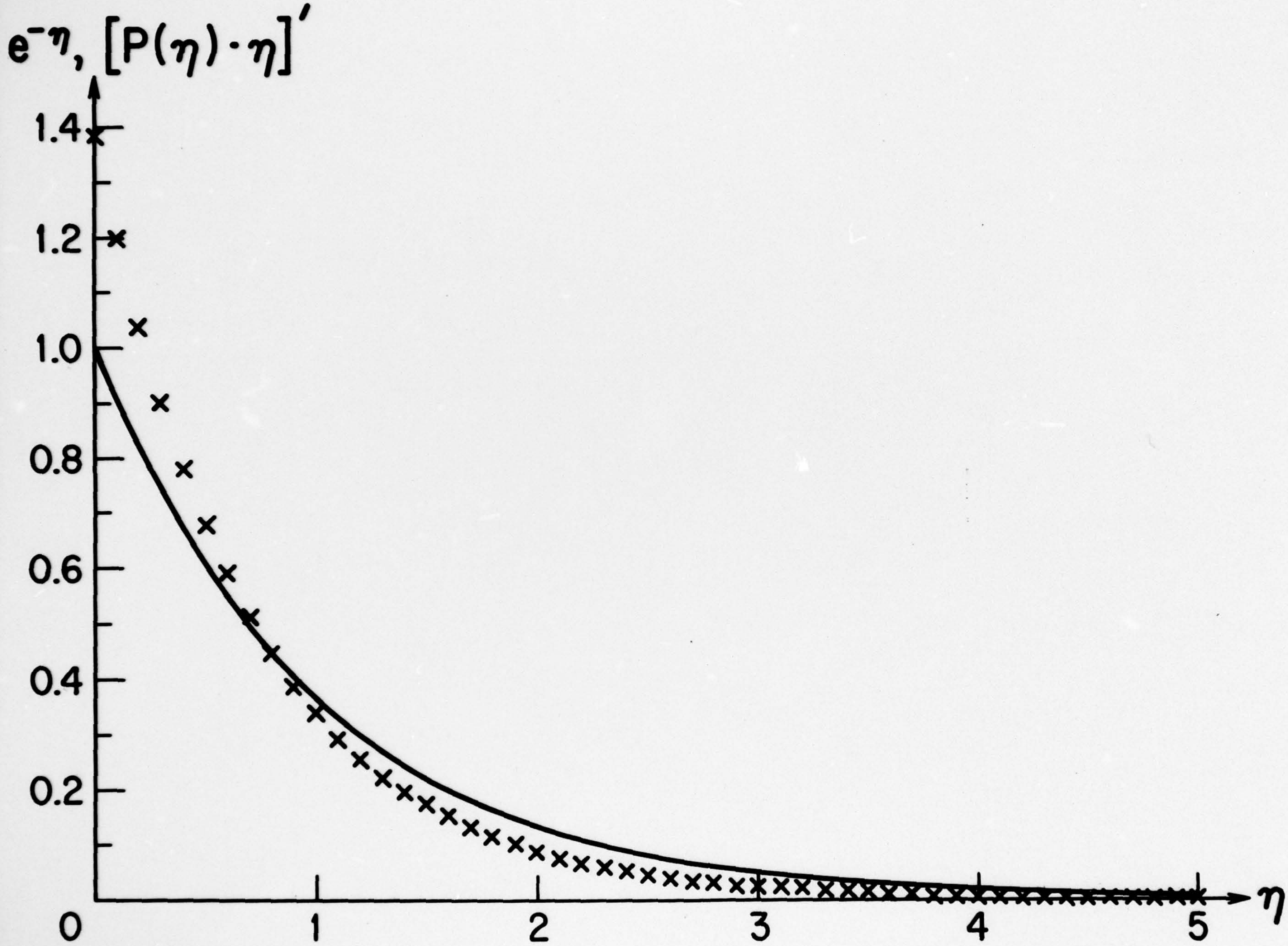

Figure 4

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Aircraft Vortices

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